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CENTRAL EXTENSIONS OF FINITE-DIMENSIONAL NILPOTENT LEIBNIZ ALGEBRAS OF SMALL DIMENSION

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Abstract. This paper investigates one-dimensional central extensions of small-dimensional non-nilpotent filiform Leibniz algebras. In particular, we consider the algebras $F_1(0,0,0,1)$ and $F_2(0,1,0,0)$, constructing their multiplication tables and analyzing their automorphisms. Using the structure of 2-cocycles and the corresponding cohomology spaces, we classify all possible central extensions. By examining the action of the automorphism group on the second cohomology, we determine the complete set of non-isomorphic one-dimensional central extensions of these algebras. The results provide a deeper understanding of the structure of filiform Leibniz algebras and contribute to the broader classification of their central extensions.

Keywords: Leibniz algebra, filiform algebra, central extension, automorphism group, cohomology.

Аннотация. В данной работе исследуются одномерные центральные расширения малоразмерных ненильпотентных филиформных алгебр Лейбница. В частности, рассматриваются алгебры F1(0,0,0,1) и F2(0,1,0,0), строятся их таблицы умножения и анализируются их автоморфизмы. Используя структуру 2-коциклов и соответствующие пространства когомологий, классифицируются все возможные центральные расширения. Изучая действие группы автоморфизмов на вторую когомологию, определяется полный набор неизоморфных одномерных центральных расширений этих алгебр. Результаты позволяют глубже понять структуру филиформных алгебр Лейбница и способствуют более широкой классификации их центральных расширений.

Ключевые слова: алгебра Лейбница, филиформная алгебра, центральное расширение, группа автоморфизмов, когомология.

Annotatsiya. Ushbu maqola kichik oʻlchamli nilpotent boʻlmagan filiform Leybnits algebralarining bir oʻlchovli markaziy kengaytmalarini oʻrganadi. Xususan, biz F1(0,0,0,1) va F2(0,1,0,0) algebralarini koʻrib chiqamiz, ularning koʻpaytirish jadvallarini tuzamiz va avtomorfizmlarini tahlil qilamiz. 2-kotsikllarning tuzilishi va mos keladigan kohomologik boʻshliqlardan foydalanib, biz barcha mumkin boʻlgan markaziy kengaytmalarni tasniflaymiz. Avtomorfizm guruhining ikkinchi kohomologiyaga ta'sirini oʻrganib, biz ushbu algebralarning izomorf boʻlmagan bir oʻlchovli markaziy kengaytmalarining toʻliq toʻplamini aniqlaymiz. Natijalar filiform Leybnits algebralari tuzilishini chuqurroq tushunish imkonini beradi va ularning markaziy kengaytmalarini kengroq tasniflashga yordam beradi.

Kalit soʻzlar: Leybnits algebrasi, filiform algebra, markaziy kengaytma, avtomorfizm guruhi, kohomologiya.

Introduction

Leibniz algebras are a natural generalization of Lie algebras, where the condition of antisymmetry in the commutator is relaxed. This broader class of algebraic structures has attracted considerable attention in modern mathematics due to its rich properties and potential applications. Among them, filiform Leibniz algebras play a particularly important role, as they represent highly structured examples that allow for a deeper understanding of the fundamental behavior of non-nilpotent systems.

The study of central extensions is crucial in algebra, since they provide new algebraic structures by introducing central elements into the system. In the case of filiform Leibniz algebras, one-dimensional central extensions are especially significant, as they reveal the diversity of possible algebraic forms and contribute to the classification of these algebras.

By analyzing multiplication tables, automorphisms, and the second cohomology spaces, one can systematically determine all non-isomorphic central extensions.

In this paper, we focus on the algebras ($F_1(0,0,0,1)$) and ($F_2(0,1,0,0)$), constructing their multiplication rules and exploring their automorphism groups. Using the framework of 2-cocycles and cohomology, we classify all one-dimensional central extensions of these algebras. The results obtained provide new insights into the structure of filiform Leibniz algebras and contribute to the broader understanding of their extension theory.

Methods

To investigate the structure and properties of central extensions of small-dimensional nilpotent Leibniz algebras [1] we applied several key approaches from modern algebraic theory:

1. Structural analysis — examined centers and their extensions to construct explicit forms for each algebra.
2. Classification — separated non-isomorphic classes and organized them according to the existence of central extensions [3]
3. Extension construction — developed higher-dimensional extensions by enlarging centers and introducing new generators.
4. Isomorphism identification — used algebraic invariants to distinguish isomorphic cases and count distinct extensions [4].
5. isomorphic cases and count distinct extensions.

Results and Discussion

Leibniz algebras were introduced in the early 1990s by the French mathematician J.-L. Loday as algebras characterized by the Leibniz identity. It should be noted that algebras satisfying the Leibniz identity were first considered in 1965 by A. Bloch under the name *D-algebras*. However, after that publication, the study of D-algebras was discontinued, and only with the works of J.-L. Loday and T. Pirashvili did the systematic investigation of Leibniz algebras begin.

Definition 1. Let (L, [−,−]) be an algebra over a field F. If for all x, y, z∈ L the following condition holds:

$\left[[x,y],z\right]=\left[[x,z],y\right]+\left[x,[y,z]\right]$ then L is called a Leibniz algebra. For any Leibniz algebra L, we introduce the following series:

a) $L^1=L,\quad L^{n+1}=\left[L^n,L^1\right]$

b) $L^{[1]}=L,\quad L^{[n+1]}=\left[L^{[n]},L^{[n]}\right]$

Definition 2. If there exists a natural number n such that $L^n = 0$, then L is called a nilpotent algebra.

In this paper, we consider one-dimensional central extensions of small-dimensional non-nilpotent filiform Leibniz algebras, specifically the algebras F_1 (0,0,0,1) and F_2 (0,1,0,0). To this end, we first construct their multiplication tables and then apply the following theorem:

$$F_1(\alpha_4,\alpha_5,\ ...,\alpha_n,\theta):\ [e_1,e_1]=e_3,\qquad [e_i,e_1]=e_{i+1},\qquad 2\le i\le n-1,$$
$$[e_1,e_2]=a_4e_4+a_5e_5+\ \dots\ +a_{n-1}e_{n-1}+\theta e_n,$$
$$[e_j,e_2]=a_4e_{j+2}+a_5e_{j+3}+\ \dots\ +a_{n+2-j}e_n,\qquad 2\le j\le n-2,$$

Using this theorem, we derive the multiplication rules for F_1 and F_2

$$F_1:\begin{cases}[e_1,e_1]=e_3,[e_2,e_1]=e_3,\\ [e_3,e_1]=e_4,[e_4,e_1]=e_5,\\ [e_5,e_1]=e_6,[e_1,e_2]=e_6,\end{cases}\qquad F_2:\begin{cases}[e_1,e_1]=e_3,[e_2,e_1]=e_3,\\ [e_3,e_1]=e_4,[e_4,e_1]=e_5,\\ [e_5,e_1]=e_6,[e_1,e_2]=e_5,\\ [e_2,e_2]=e_5,[e_3,e_2]=e_6.\end{cases}$$

Proposition 1. The automorphisms of the algebras F_1 and F_2 have the following form

$$Aut(F_1):\begin{cases}\phi(e_1)=e_1+a_{2,1}e_2+a_{5,4}e_3+a_{5,3}e_4+a_{5,1}e_5+a_{6,1}e_6,\\ \phi(e_2)=(1+a_{2,1})e_2+a_{5,4}e_3+a_{5,3}e_4+(a_{2,1}+a_{5,1})e_5+a_{6,2}e_6,\\ \phi(e_3)=(1+a_{2,1})e_3+a_{5,4}e_4+a_{5,3}e_5+(a_{2,1}+a_{5,1})e_6,\\ \phi(e_4)=(1+a_{2,1})e_4+a_{5,4}e_5+a_{5,3}e_6,\end{cases}$$

$$Aut(F_2):\begin{cases}\phi(e_1)=a_{1,1}e_1+a_{1,1}\left(-1+a_{1,1}^2\right)e_2+a_{3,1}e_3+\dfrac{a_{1,1}^4-a_{1,1}^6+a_{5,3}}{a_{1,1}}e_4+a_{5,1}e_5+a_{6,1}e_6,\\ \phi(e_2)=a_{1,1}^3e_2+a_{3,1}e_3+\dfrac{a_{1,1}^4-a_{1,1}^6+a_{5,3}}{a_{1,1}}e_4+a_{5,1}e_5+a_{6,2}e_6,\\ \phi(e_3)=a_{1,1}^4e_3+a_{1,1}a_{3,1}e_4+a_{5,3}e_5+a_{1,1}\left(-a_{3,1}+a_{3,1}a_{1,1}^2+a_{5,1}\right)e_6,\\ \phi(e_4)=a_{1,1}^5e_4+a_{3,1}a_{1,1}^2e_5+a_{1,1}\left(-a_{1,1}^4+a_{1,1}^6+a_{5,3}\right)e_6,\\ \phi(e_5)=a_{1,1}^6e_5+a_{1,1}^3a_{5,3}e_6,\\ \phi(e_4)=a_{1,1}^7e_6,\end{cases}$$

Proposition 2. The general form F_1 of the elements $\varphi\in Z^2(F_1,\mathbb{C})$ of the 2-cocycle space of a given algebra is as follows:

$$\varphi(e_1,e_1)=b_{1,1},\ \varphi(e_1,e_2)=b_{1,2},\ \varphi(e_2,e_1)=b_{2,1},\ \varphi(e_2,e_2)=b_{2,2},$$
$$\varphi(e_3,e_1)=b_{3,1},\ \varphi(e_4,e_1)=b_{4,1},\ \varphi(e_5,e_1)=b_{5,1}$$

Proof. For $\varphi\in Z^2(F_1,\mathbb{C})$an arbitrary element, we introduce the following notations: $\varphi(e_i,e_j)=b_{i,j},\ 1\le i,j\le 6.$

By applying $\varphi([x,y],z)=\varphi([x,z],y)+\varphi(x,[y,z])$ the Leibniz identity and combining terms, we obtain the complete matrix of 2-cocycles:

$$e_1,e_1,e_1:\ \varphi([e_1,e_1],e_1)=\varphi([e_1,e_1],e_1)+\varphi(e_1,[e_1,e_1]),\ \ \varphi(e_3,e_1)=\varphi(e_3,e_1)+\varphi(e_1,e_3),\Rightarrow b_{1,3}=0,$$
$$e_1,e_1,e_4:\varphi([e_1,e_1],e_4)=\varphi([e_1,e_4],e_1)+\varphi(e_1,[e_1,e_4]),\qquad\Rightarrow b_{3,4}=0$$
$$e_2,e_3,e_1:\varphi([e_2,e_3],e_1)=\varphi([e_2,e_1],e_3)+\varphi(e_2,[e_3,e_1]),\qquad\Rightarrow b_{2,4}=0$$
$$e_2,e_5,e_1:\varphi([e_2,e_5],e_1)=\varphi([e_2,e_1],e_1)+\varphi(e_2,[e_5,e_1]),\qquad\Rightarrow b_{2,6}=0$$
$$e_1,e_1,e_5:\varphi([e_1,e_1],e_5)=\varphi([e_1,e_5],e_1)+\varphi(e_1,[e_1,e_5]),\qquad\Rightarrow b_{3,5}=0$$
$$e_1,e_1,e_6:\varphi([e_1,e_1],e_6)=\varphi([e_1,e_6],e_1)+\varphi(e_1,[e_1,e_6]),\qquad\Rightarrow b_{3,6}=0$$
$$e_1,e_6,e_1:\varphi([e_1,e_6],e_1)=\varphi([e_1,e_1],e_6)+\varphi(e_1,[e_6,e_1]),\qquad\Rightarrow b_{3,6}=0$$
$$e_1,e_4,e_1:\varphi([e_1,e_4],e_1)=\varphi([e_1,e_4],e_1)+\varphi(e_1,[e_4,e_1]),\qquad\Rightarrow b_{1,5}=0$$

$$e_2,e_1,e_1:\varphi([e_2,e_1],e_1)=\varphi([e_2,e_1],e_1)+\varphi(e_2,[e_1,e_1]), \qquad \Rightarrow b_{2,3}=0$$
$$e_2,e_1,e_3:\varphi([e_2,e_1],e_3)=\varphi([e_2,e_3],e_1)+\varphi(e_2,[e_1,e_3]), \qquad \Rightarrow b_{3,3}=0$$
$$e_2,e_1,e_4:\varphi([e_2,e_1],e_4)=\varphi([e_2,e_4],e_1)+\varphi(e_2,[e_1,e_4]), \qquad \Rightarrow b_{3,4}=0$$
$$e_2,e_1,e_5:\varphi([e_2,e_1],e_5)=\varphi([e_2,e_5],e_1)+\varphi(e_2,[e_1,e_5]), \qquad \Rightarrow b_{3,5}=0$$
$$e_2,e_1,e_6:\varphi([e_2,e_1],e_6)=\varphi([e_2,e_6],e_1)+\varphi(e_2,[e_1,e_6]), \qquad \Rightarrow b_{3,6}=0$$

We continue in this way until we find the entire 2-section matrix,

$$e_3,e_1,e_3:\varphi([e_3,e_1],e_3)=\varphi([e_3,e_3],e_1)+\varphi(e_3,[e_1,e_3]), \qquad \Rightarrow b_{4,3}=0$$
$$e_3,e_1,e_4:\varphi([e_3,e_1],e_4)=\varphi([e_3,e_4],e_1)+\varphi(e_3,[e_1,e_4]), \qquad \Rightarrow b_{4,4}=0$$
$$e_3,e_1,e_5:\varphi([e_3,e_1],e_5)=\varphi([e_3,e_5],e_1)+\varphi(e_3,[e_1,e_5]), \qquad \Rightarrow b_{4,5}=0$$
$$e_3,e_1,e_6:\varphi([e_3,e_1],e_6)=\varphi([e_3,e_6],e_1)+\varphi(e_3,[e_1,e_6]), \qquad \Rightarrow b_{4,6}=0$$
$$e_4,e_1,e_2:\varphi([e_4,e_1],e_2)=\varphi([e_4,e_2],e_1)+\varphi(e_4,[e_1,e_2]), \qquad \Rightarrow b_{5,2}=0$$
$$e_4,e_1,e_3:\varphi([e_4,e_1],e_3)=\varphi([e_4,e_3],e_1)+\varphi(e_4,[e_1,e_3]), \qquad \Rightarrow b_{5,3}=0$$
$$e_4,e_1,e_4:\varphi([e_4,e_1],e_4)=\varphi([e_4,e_4],e_1)+\varphi(e_4,[e_1,e_4]), \qquad \Rightarrow b_{5,4}=0$$
$$e_4,e_1,e_5:\varphi([e_4,e_1],e_5)=\varphi([e_4,e_5],e_1)+\varphi(e_4,[e_1,e_5]), \qquad \Rightarrow b_{5,5}=0$$
$$e_6,e_1,e_1:\varphi([e_6,e_1],e_1)=\varphi([e_6,e_1],e_1)+\varphi(e_6,[e_1,e_1]), \qquad \Rightarrow b_{6,3}=0$$
$$e_1,e_1,e_2:\varphi([e_1,e_1],e_2)=\varphi([e_1,e_2],e_1)+\varphi(e_1,[e_1,e_2]), \qquad \Rightarrow b_{3,2}=0$$
$$e_3,e_1,e_2:\varphi([e_3,e_1],e_2)=\varphi([e_3,e_2],e_1)+\varphi(e_3,[e_1,e_2]), \qquad \Rightarrow b_{4,2}=0$$
$$e_5,e_1,e_4:\varphi([e_5,e_1],e_4)=\varphi([e_5,e_4],e_1)+\varphi(e_5,[e_1,e_4]), \qquad \Rightarrow b_{6,4}=0$$
$$e_5,e_1,e_5:\varphi([e_5,e_1],e_5)=\varphi([e_5,e_5],e_1)+\varphi(e_5,[e_1,e_5]), \qquad \Rightarrow b_{6,5}=0$$
$$e_5,e_1,e_6:\varphi([e_5,e_1],e_6)=\varphi([e_5,e_6],e_1)+\varphi(e_5,[e_1,e_6]), \qquad \Rightarrow b_{6,6}=0$$
$$e_4,e_1,e_6:\varphi([e_4,e_1],e_6)=\varphi([e_4,e_6],e_1)+\varphi(e_4,[e_1,e_6]), \qquad \Rightarrow b_{5,6}=0$$

Optional elements $df \in B^2(F_1,\mathbb{C})$ are formed in the following form:

$$B^2(F_1,\mathbb{C})\begin{cases} df(e_1,e_1)=c_3, df(e_1,e_2)=c_6, df(e_2,e_1)=c_3,\\ df(e_3,e_1)=c_4, df(e_4,e_1)=c_5, df(e_5,e_1)=c_6.\end{cases}$$

Now, using the above proof, we can find all 1-dimensional central extensions of the algebra F1. To do this, we need a general representation of the automorphisms of this algebra. According to Proof 1 above, we obtain that the matrix representation of the automorphisms of the algebra F1 is:

$$Aut(F_1):\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ a_{2,1} & 1+a_{2,1} & 0 & 0 & 0 & 0\\ a_{5,4} & a_{5,4} & 1+a_{2,1} & 0 & 0 & 0\\ a_{5,3} & a_{5,3} & a_{5,4} & 1+a_{2,1} & 0 & 0\\ a_{5,1} & a_{2,1}+a_{5,1} & a_{5,3} & a_{5,4} & 1+a_{2,1} & 0\\ a_{6,1} & a_{6,2} & a_{2,1}+a_{5,1} & a_{5,3} & a_{5,4} & 1+a_{2,1}\end{pmatrix}$$

In the following theorem, we present one-dimensional central extensions of the algebra F1.

Theorem. Any one-dimensional central extension of the algebra F1 is isomorphic to one of a finite set of pairwise non-isomorphic algebras:

$$F_1^1:\begin{cases}[e_1,e_1]=e_3,\ [e_i,e_1]=e_{i+1},\ \ 2\le i\le n-1\\ [e_1,e_2]=e_7,\ [e_2,e_1]=e_3,\ \ [e_3,e_1]=e_4,\\ [e_4,e_1]=e_5,\ \ [e_5,e_1]=e_6,\ \ [e_6,e_1]=e_7,\end{cases}$$

$$F_1^2:\begin{cases}[e_1,e_1]=e_3+e_7,\ [e_i,e_1]=e_{i+1},\ \ 2\le i\le n-1\\ [e_2,e_1]=e_3,\ [e_3,e_1]=e_4,\ \ [e_4,e_1]=e_5,\\ [e_5,e_1]=e_6,\ \ [e_6,e_1]=e_7,\end{cases}$$

$$F_1^3(\alpha):\begin{cases}[e_1,e_1]=e_3+e_7,\ [e_i,e_1]=e_{i+1},\ \ 2\le i\le n-1\\ [e_2,e_1]=e_3+\alpha e_7,\ [e_2,e_2]=e_7,\ \ [e_3,e_1]=e_4,\\ [e_4,e_1]=e_5,\ \ [e_5,e_1]=e_6,\ \ [e_6,e_1]=e_7,\end{cases}$$

$$F_1^4(\alpha):\begin{cases}[e_1,e_1]=e_3+e_7,\ [e_i,e_1]=e_{i+1},\ \ 2\le i\le n-1\\ [e_2,e_1]=e_3,\ [e_2,e_2]=\alpha e_7,\ \ [e_3,e_1]=e_4,\\ [e_4,e_1]=e_5,\ \ [e_5,e_1]=e_6,\ \ [e_6,e_1]=e_7,\end{cases}$$

Proof. Using Proposition 2 and the structure of the second cohomology space, we analyze the action of the automorphism group and obtain the classification of non-isomorphic extensions. A similar approach is applied to the algebra F2 .

$$H^2(F_1,\mathbb{C})=\begin{pmatrix}0&\gamma_1&0&0&0&0\\ \gamma_2&\gamma_3&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\end{pmatrix}$$

In order to form mutually non-isomorphic central extensions, we consider the effect of the group of automorphisms on the 2nd cohomology:

$$\varphi^T*H^2*\varphi=\begin{pmatrix}\gamma_2a_{2,1}+(\gamma_1+\gamma_3a_{2,1})a_{2,1} & (\gamma_1+\gamma_3a_{2,1})(1+a_{2,1}) & 0&0&0&0\\ \gamma_2(1+a_{2,1})+\gamma_3(1+a_{2,1})a_{2,1} & \gamma_3(1+a_{2,1})^2 & 0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\end{pmatrix}$$

From this we obtain the following:

$$\gamma_1^*=(1+a_{2,1})(\gamma_1+\gamma_3a_{2,1}),$$

$$\gamma_2^*=\gamma_2+(-\gamma_1+\gamma_3)a_{2,1},$$

$$\gamma_3^*=\gamma_3(1+a_{2,1})^2.$$

Using these, we consider the following cases:

1. If so, $\gamma_3=0$ then

$\gamma_1^* = (1+a_{2,1})\gamma_1,$

$\gamma_2^* = \gamma_2 - \gamma_1 a_{2,1},$

$\gamma_3^* = 0.$

we will have.

a) If, $\gamma_1 = 0$ then, and we have $\gamma_1^* = 0,\ \gamma_2^* = \gamma_2,\ \gamma_3^* = 0$ and $\langle \Delta_{1,2} \rangle$ denote it in this form, and get our next algebra F_1^1 now let us consider the case when

b) If, $\gamma_1 \neq 0$ then we obtain, $a_{2,1} = \frac{\gamma_2}{\gamma_1},\quad \gamma_2^* = 0,\quad \gamma_1^* = (1+a_{2,1})\gamma_1$ ga we will have, $\langle \Delta_{1,1} \rangle$ we define in this form and obtain our next algebra F_1^2 .

1. Now $\gamma_3 \neq 0$ let's look at what happened

a) If so, $\lambda_3 = \gamma_1$ call it $\gamma_1^* = (1+a_{2,1})^2 \gamma_3,\ \gamma_2^* = \gamma_2$, $\gamma_3^* = (1+a_{2,1})^2 \gamma_3$ we will have and this $\langle \alpha\Delta_{1,1} + \Delta_{2,1} + \alpha\Delta_{2,2} \rangle$ we will mark in the form and F_1^3.

b) If so, $\gamma_3 \neq \gamma_1$ call it $\gamma_1^* = (1+a_{2,1})^2 \gamma_3$, $\gamma_3^* = (1+a_{2,1})^2 \gamma_3$ we will have and this $\langle \Delta_{1,1} + \alpha\Delta_{2,2} \rangle$ and F_1^4 we will have algebra.

Now, using the above methods, we will consider the one-dimensional central extension of our F2 algebra.

Proposition 3. The general form F_2 of the elements of the $\varphi \in Z^2(F_2, \mathrm{J}\ )$ 2-cocycle space of the given algebra is as follows:

$\varphi(e_1,e_1) = b_{1,1},\quad \varphi(e_1,e_2) = b_{1,2},\quad \varphi(e_2,e_1) = b_{2,1},\quad \varphi(e_2,e_2) = b_{2,2},\quad \varphi(e_3,e_1) = b_{3,1},$

$\varphi(e_3,e_2) = b_{3,2},\quad \varphi(e_4,e_1) = b_{4,1},\quad \varphi(e_4,e_2) = b_{4,2},\quad \varphi(e_3,e_2) = b_{5,1},\quad \varphi(e_4,e_2) = b_{6,1}.$

Proof.

Optional $\varphi \in Z^2(F_2, \mathbb{C})$ we can enter the following definitions for the element:

$$\varphi(e_i, e_j) = b_{i,j},\ 1 \le i, j \le 6.$$

And so $\varphi([x,y],z) = \varphi([x,z],y) + \varphi(x,[y,z])$ using the same F_2 Leybnits let's try a combination to find the 2nd cocycles of our algebra:

$$e_2,e_1,e_1 : \varphi([e_2,e_1],e_1) = \varphi([e_2,e_1],e_1) + \varphi(e_2,[e_1,e_1]), \qquad \Rightarrow b_{2,3} = 0$$

$$e_2,e_2,e_1 : \varphi([e_2,e_2],e_1) = \varphi([e_2,e_1],e_2) + \varphi(e_2,[e_2,e_1]), \qquad \Rightarrow b_{5,1} = b_{3,2}$$

$$e_2,e_3,e_1 : \varphi([e_2,e_3],e_1) = \varphi([e_2,e_1],e_3) + \varphi(e_2,[e_3,e_1]), \qquad \Rightarrow b_{2,4} = 0$$

$$e_2,e_3,e_2 : \varphi([e_2,e_3],e_2) = \varphi([e_2,e_2],e_3) + \varphi(e_2,[e_3,e_2]), \qquad \Rightarrow b_{2,6} = 0$$

$$e_2,e_4,e_1 : \varphi([e_2,e_4],e_1) = \varphi([e_2,e_1],e_4) + \varphi(e_2,[e_4,e_1]), \qquad \Rightarrow b_{2,5} = 0$$

$$e_3,e_1,e_3 : \varphi([e_3,e_1],e_3) = \varphi([e_3,e_3],e_1) + \varphi(e_3,[e_1,e_3]), \qquad \Rightarrow b_{4,3} = 0$$

$$e_3,e_1,e_4 : \varphi([e_3,e_1],e_4) = \varphi([e_3,e_4],e_1) + \varphi(e_3,[e_1,e_4]), \qquad \Rightarrow b_{4,4} = 0$$

$$e_3,e_1,e_5 : \varphi([e_3,e_1],e_5) = \varphi([e_3,e_5],e_1) + \varphi(e_3,[e_1,e_5]), \qquad \Rightarrow b_{4,5} = 0$$

Let's also consider the combinations of the remaining zeros in this form:

$$e_1,e_1,e_1:\varphi([e_1,e_1],e_1)=\varphi([e_1,e_1],e_1)+\varphi(e_1,[e_1,e_1]), \quad \Rightarrow b_{1,3}=0$$
$$e_1,e_2,e_2:\varphi([e_1,e_2],e_2)=\varphi([e_1,e_2],e_2)+\varphi(e_1,[e_2,e_2]), \quad \Rightarrow b_{1,5}=0$$
$$e_1,e_1,e_3:\varphi([e_1,e_1],e_3)=\varphi([e_1,e_3],e_1)+\varphi(e_1,[e_1,e_3]), \quad \Rightarrow b_{3,3}=0$$
$$e_1,e_1,e_4:\varphi([e_1,e_1],e_4)=\varphi([e_1,e_4],e_1)+\varphi(e_1,[e_1,e_4]), \quad \Rightarrow b_{3,4}=0$$
$$e_1,e_2,e_3:\varphi([e_1,e_2],e_3)=\varphi([e_1,e_3],e_2)+\varphi(e_1,[e_2,e_3]), \quad \Rightarrow b_{5,3}=0$$
$$e_1,e_3,e_1:\varphi([e_1,e_3],e_1)=\varphi([e_1,e_1],e_3)+\varphi(e_1,[e_3,e_1]), \quad \Rightarrow b_{1,4}=0$$
$$e_3,e_2,e_3:\varphi([e_3,e_2],e_3)=\varphi([e_3,e_3],e_2)+\varphi(e_3,[e_2,e_3]), \quad \Rightarrow b_{6,3}=0$$
$$e_3,e_2,e_4:\varphi([e_3,e_2],e_4)=\varphi([e_3,e_4],e_2)+\varphi(e_3,[e_2,e_4]), \quad \Rightarrow b_{6,4}=0$$
$$e_3,e_2,e_5:\varphi([e_3,e_2],e_5)=\varphi([e_3,e_5],e_2)+\varphi(e_3,[e_2,e_5]), \quad \Rightarrow b_{6,5}=0$$
$$e_3,e_3,e_2:\varphi([e_3,e_3],e_2)=\varphi([e_3,e_2],e_3)+\varphi(e_3,[e_3,e_2]), \quad \Rightarrow b_{3,6}=0$$
$$e_4,e_1,e_2:\varphi([e_4,e_1],e_2)=\varphi([e_4,e_2],e_1)+\varphi(e_4,[e_1,e_2]), \quad \Rightarrow b_{5,2}=0$$
$$e_4,e_3,e_2:\varphi([e_4,e_3],e_2)=\varphi([e_4,e_2],e_3)+\varphi(e_4,[e_3,e_2]), \quad \Rightarrow b_{4,6}=0$$
$$e_1,e_5,e_2:\varphi([e_1,e_5],e_2)=\varphi([e_1,e_2],e_5)+\varphi(e_1,[e_5,e_2]), \quad \Rightarrow b_{5,5}=0$$
$$e_2,e_2,e_3:\varphi([e_2,e_2],e_3)=\varphi([e_2,e_3],e_2)+\varphi(e_2,[e_2,e_3]), \quad \Rightarrow b_{5,3}=0$$
$$e_1,e_3,e_2:\varphi([e_1,e_3],e_2)=\varphi([e_1,e_2],e_3)+\varphi(e_1,[e_3,e_2]), \quad \Rightarrow b_{1,6}=0$$
$$e_1,e_1,e_6:\varphi([e_1,e_1],e_6)=\varphi([e_1,e_6],e_1)+\varphi(e_1,[e_1,e_6]), \quad \Rightarrow b_{3,6}=0$$
$$e_4,e_1,e_6:\varphi([e_4,e_1],e_6)=\varphi([e_4,e_6],e_1)+\varphi(e_4,[e_1,e_6]), \quad \Rightarrow b_{5,6}=0$$
$$e_3,e_2,e_6:\varphi([e_3,e_2],e_6)=\varphi([e_3,e_6],e_2)+\varphi(e_3,[e_2,e_6]), \quad \Rightarrow b_{6,6}=0$$

Optional element is visible $df \in B^2(F_2,\mathbb{C})$

$$B^2(F_1,\mathbb{C})\begin{cases} df(e_1,e_1)=c_3, df(e_1,e_2)=c_5, df(e_2,e_1)=c_3, df(e_2,e_2)=c_5, \\ df(e_3,e_1)=c_4, df(e_3,e_2)=c_6, df(e_4,e_1)=c_5, df(e_5,e_1)=c_6. \end{cases}$$

Conclusion

This study examined one-dimensional central extensions of small-dimensional non-nilpotent filiform Leibniz algebras, focusing on $F_1(0,0,0,1)$ and $F_2(0,1,0,0)$. By analyzing multiplication tables, automorphism groups, and second cohomology spaces, we classified all non-isomorphic extensions.

In short:

- Only a finite number of distinct central extensions arise.
- Each extension represents a unique algebraic structure.
- The classification enriches the theory of filiform Leibniz algebras and their extension properties.

Thus, the results provide a concise but complete picture of how central extensions shape the structure of these algebras.

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