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# WAVE PROPAGATION IN A COSSERAT ELASTIC HALF-SPACE: DISPERSION ANALYSIS OF SURFACE AND BULK WAVES

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***Abstract:*** The paper investigates the propagation of harmonic waves in an elastic half-space whose material is described by a simplified two-dimensional Cosserat continuum (micropolar medium). The equations of motion of the Cosserat medium are reduced, by means of the Green–Lamb (scalar–vector potential) transformation, to a system of ordinary differential equations with complex coefficients. The complex character of the coefficients is due to the viscoelastic properties of the medium, modelled by the three-parameter weakly singular relaxation kernel of Rzhanitsyn–Koltunov. The general solution of the system is constructed, volume wave velocities are determined, and a dispersion equation for a layered Cosserat structure is derived. It is shown that a Rayleigh-type surface wave in a Cosserat half-space is dispersive, in contrast to the classical case: the "phase velocity–frequency" plane contains two dispersion branches, a lower ("acoustic") and an upper ("optical") one. The phase velocity of the surface wave exceeds the phase velocity of the bulk displacement wave over the whole frequency range; the wave propagates in the horizontal direction and decays with depth. From the same system, solutions describing Lamb waves (symmetric and antisymmetric plate modes) and Stoneley waves at the interface of two half-spaces are obtained. Numerical results show that the normal stress moment in the Cosserat medium exceeds the classical value by approximately 24 %, and the displacement-amplitude trajectory is elliptical.

***Keywords****:* Cosserat continuum; micropolar medium; surface wave; Rayleigh wave;dispersion equation; viscoelasticity; relaxation kernel; elastic half-space

***Annotatsiya***. Maqolada materiali soddalashtirilgan ikki o'lchovli Cosserat kontinuum (mikropolyar muhit) bilan tasvirlangan elastik yarim fazoda garmonik to'lqinlarning tarqalishi o'rganiladi. Kosserat muhitining harakat tenglamalari Green-Lamb (skalar-vektor potensiali) transformatsiyasi yordamida murakkab koeffitsientli oddiy differensial tenglamalar tizimiga keltiriladi. Koeffitsientlarning murakkab xarakteri Rjanitsyn-Koltunovning uch parametrli zaif yagona relaksatsiya yadrosi tomonidan modellashtirilgan muhitning viskoelastik xususiyatlariga bog'liq. Tizimning umumiy yechimi quriladi, hajm to‘lqin tezligi aniqlanadi va qatlamli Kosserat strukturasi uchun dispersiya tenglamasi olinadi. Kosserat yarim fazosida Rayleigh tipidagi sirt to'lqini klassik holatdan farqli ravishda dispersiv ekanligi ko'rsatilgan: "faza tezligi-chastota" tekisligi ikkita dispersiya shoxlarini o'z ichiga oladi, pastki ("akustik") va yuqori ("optik"). Sirt to'lqinining faza tezligi butun chastota diapazoni bo'ylab ommaviy siljish to'lqinining faza tezligidan oshadi; to'lqin gorizontal yo'nalishda tarqaladi va chuqurlik bilan parchalanadi. Xuddi shu tizimdan ikkita yarim bo'shliqning interfeysida Qo'zi to'lqinlarini (simmetrik va antisimmetrik plastinka rejimlari) va Stoneley to'lqinlarini tavsiflovchi echimlar olinadi. Raqamli natijalar shuni ko'rsatadiki, Kosserat muhitidagi normal kuchlanish momenti klassik qiymatdan taxminan 24% ga oshadi va siljish-amplitudali traektoriya elliptikdir.

**Kalit so'zlar:** Cosserat kontinuum; mikropolyar muhit; sirt to'lqini; Rayleigh to'lqini;dispersiya tenglamasi; yopishqoqlik; dam olish yadrosi; elastik yarim bo'shliq

***Аннотация:*** В статье исследуется распространение гармонических волн в упругом полупространстве, материал которого описывается упрощенным двумерным континуумом Коссера (микрополярной средой). Уравнения движения среды Коссера сводятся, посредством преобразования Грина–Лэмба (скалярно-векторный потенциал), к системе обыкновенных дифференциальных

уравнений с комплексными коэффициентами. Комплексный характер коэффициентов обусловлен вязкоупругими свойствами среды, моделируемыми трехпараметрическим слабосингулярным ядром релаксации Ржаницына–Колтунова. Построено общее решение системы, определены объемные скорости волн и выведено дисперсионное уравнение для слоистой структуры Коссера. Показано, что поверхностная волна типа Рэлея в полупространстве Коссера является дисперсионной, в отличие от классического случая: плоскость «фазовая скорость–частота» содержит две дисперсионные ветви, нижнюю («акустическую») и верхнюю («оптическую»). Фазовая скорость поверхностной волны превышает фазовую скорость объемной волны смещения во всем частотном диапазоне; волна распространяется в горизонтальном направлении и затухает с глубиной. Из той же системы получены решения, описывающие волны Лэмба (симметричные и антисимметричные моды пластин) и волны Стоунли на границе двух полупространств. Численные результаты показывают, что момент нормального напряжения в среде Коссера превышает классическое значение примерно на 24 %, а траектория амплитуды смещения имеет эллиптическую форму.

***Ключевые слова:*** континуум Коссера; микрополярная среда; поверхностная волна; волна Рэлея; дисперсионное уравнение; вязкоупругость; ядро релаксации; упругое полупространство

## Introduction

Classical elasticity theory does not describe size effects observed in materials with a pronounced microstructure (granular and composite materials, polycrystals, foams, masonry, bone). The Cosserat (micropolar) continuum model, in which each material point possesses, in addition to the displacement vector, an independent rotation vector, removes this shortcoming and is now widely used in seismology, geomechanics and non-destructive testing.

The aim of the present work is to derive and analyse the dispersion relations for free and forced harmonic wave propagation in an elastic half-space made of a Cosserat material, and to construct a general numerical algorithm for layered.

## Materials and methods

For harmonic wave propagation in the $n$-th layer of a Cosserat medium the equations of motion take the form [1-3]:

$$\left(2\bar{\mu}_n+\bar{\lambda}_n\right)\mathrm{graddiv}\vec{U}_n-\left(\bar{\mu}_n+\bar{\alpha}_n\right)\mathrm{rotrot}\vec{U}_n+2\bar{\alpha}_n\,\mathrm{rot}\vec{\Omega}_n+\rho_n\omega^2\vec{U}_n(z)=0, \quad (1)$$

$$\left(2\bar{\gamma}_n+\bar{\beta}_n\right)\mathrm{graddiv}\vec{\Omega}_n-\left(\bar{\gamma}_n+\bar{\varepsilon}_n\right)\mathrm{rotrot}\vec{\Omega}_n+2\bar{\alpha}_n\,\mathrm{rot}\vec{U}_n+\left(I_n\omega^2-4\bar{\alpha}_n\right)\vec{\Omega}_n(z)=0, \quad (2)$$

where $\vec{U}_n$ is the displacement vector, $\vec{\Omega}_n$ the microrotation vector, $\rho_n$ the material density, $I_n$ the inertia characteristic of the material (moment of inertia of the microelement), $\lambda_n,\mu_n$ the Lamé constants and $\alpha_n,\beta_n,\gamma_n,\varepsilon_n$ the Cosserat (couple-stress) elastic parameters satisfying the constraints $\bar{\mu}_n>0$, $\bar{\lambda}_n+2\bar{\mu}_n>0$,

$\bar{\alpha}_n\ge 0$, $\bar{\gamma}_n+\bar{\varepsilon}_n>0$ [4-8]. The bars denote viscoelastic operators with complex values.

Reduction to a system of ordinary differential equations: The solution is sought as a sum of scalar and vector potentials (Green–Lamb transformation):

$$\vec{U}_n=\mathrm{grad}\Phi_n+\mathrm{rot}\vec{\Psi}_n,\qquad \vec{\Omega}_n=\mathrm{grad}\Sigma_n+\mathrm{rot}\vec{\Lambda}_n. \quad (3)$$

Substituting (3) into (2)–(3) yields a system of ordinary differential equations with respect to the depth coordinate $z$:

$$\Phi_{n''}(z)-\left(k^2-k_{1n}^2\right)\Phi_n=0,\qquad \Psi_{n''}(z)-\left(k^2-k_{2n}^2\right)\Psi_n=0,$$

$$\Sigma_{n''}(z)-\left(k^2-k_{3n}^2\right)\Sigma_n=0,\qquad \Lambda_{n''}(z)-\left(k^2-k_{4n}^2\right)\Lambda_n=0, \quad (4)$$

where $k$ is the wavenumber and

$$k_{1n}^2=\frac{\rho_n\omega^2}{\bar{\lambda}_n+2\bar{\mu}_n},\qquad k_{2n}^2=\frac{\rho_n\omega^2}{\bar{\mu}_n+\bar{\alpha}_n},\qquad k_{3n}^2=\frac{I_n\omega^2-4\bar{\alpha}_n}{\bar{\beta}_n+2\bar{\gamma}_n},\qquad k_{4n}^2=\frac{I_n\omega^2-4\bar{\alpha}_n}{\bar{\gamma}_n+\bar{\varepsilon}_n}. \quad (4.a)$$

Dimensionless variables: To simplify the solution and computation, all dimensional quantities are converted to dimensionless form by introducing the new variables

$$\zeta = kz, \qquad \text{ш}_{jn} = \frac{k_{jn}}{k} \quad (j = \overline{1,4}), \qquad \Omega = \frac{\omega}{\omega_*}, \tag{5}$$

where $\omega_*$ is the characteristic frequency. Substitution of (5) into (4) gives a second-order system of ordinary differential equations with complex coefficients:

$$\frac{d^2\mathbf{Y}_n}{d\zeta^2} - \mathbf{K}_n(\Omega)\mathbf{Y}_n = 0, \qquad \mathbf{Y}_n = (\Phi_n, \Psi_n, \Sigma_n, \Lambda_n)^{\mathrm{T}}. \tag{6}$$

General solution and wave velocities: The solutions of system (6) have the for

$$\mathbf{Y}_n(\zeta) = \sum_{j=1}^{4}\left(\mathbf{C}_j^- e^{-\kappa_j\zeta} + \mathbf{C}_j^+ e^{\kappa_j\zeta}\right), \qquad \kappa_j = \sqrt{1-\text{ш}_{jn}^2}, \tag{7}$$

where $C_1$ and $C_2$ are the amplitudes of the longitudinal and transverse waves, $C_4$ and $C_5$ are the Cosserat parameters, and $C_3$ is an abbreviating constant. For a half-space the solution bounded at infinity reads

$$\Phi_n = C_1 e^{-\kappa_1\zeta} + C_3 e^{\kappa_1\zeta}, \qquad \Psi_n = C_2 e^{-\kappa_2\zeta} + C_6 e^{\kappa_2\zeta}, \tag{8.a}$$

$$\Sigma_n = C_4 e^{-\kappa_3\zeta} + C_7 e^{\kappa_3\zeta}, \qquad \Lambda_n = C_5 e^{-\kappa_4\zeta} + C_8 e^{\kappa_4\zeta}. \tag{8.b}$$

The bulk wave velocities in the Cosserat medium are

$$c_{1n} = \sqrt{\frac{\bar{\lambda}_n + 2\bar{\mu}_n}{\rho_n}}, \qquad c_{2n} = \sqrt{\frac{\bar{\mu}_n + \bar{\alpha}_n}{\rho_n}}, \qquad c_{3n} = \sqrt{\frac{\bar{\beta}_n + 2\bar{\gamma}_n}{I_n}}, \qquad c_{4n} = \sqrt{\frac{\bar{\gamma}_n + \bar{\varepsilon}_n}{I_n}}. \tag{8.c}$$

Dispersion equation: From the boundary conditions of the free surface absence of force stresses and couple stresses) one obtains the dispersion equation

$$\Delta(k,\omega) \equiv \det \mathbf{T}(k,\omega) = 0, \tag{9}$$

where $\mathbf{T}$ is the boundary-condition matrix assembled from (8)? Because the Cosserat medium is viscoelastic, the unknowns in (9) are complex quantities, i.e. (2.45) is a transcendental equation with complex parameters.

It is solved numerically by the Müller method combined with the Laplace (Durand) method. The general algorithm for obtaining dispersion relations and numerical results in multilayered Cosserat structures is described in Section 4 of the

In the computations the relaxation of the medium is described by the three-parameter weakly singular Rzhanitsyn–Koltunov kernel

$$\mathrm{R}(t) = \frac{\chi}{\Gamma(1-\alpha)} t^{-\alpha} e^{-\beta t}, \qquad 0 < \alpha < 1, \quad \beta > 0, \quad \chi > 0,$$

so that $\bar{\nu}_n = \nu_n\left(1 - \text{ш}_n \mathrm{R}^*\right)$ for each elastic modulus $\nu \in \{\lambda, \mu, \alpha, \beta, \gamma, \varepsilon\}$.

Simplified two-dimensional Cosserat model: Consider the propagation of a surface wave in an elastic half-space based on the simplified two-dimensional Cosserat model:

$$\rho\ddot{u}_i = (\lambda+\mu)\partial_i(\partial_j u_j) + (\mu+\alpha)\nabla^2 u_i + 2\alpha e_{ij3}\partial_j\omega_3 + \rho f_i, \tag{10.a}$$

$$\rho j\ddot{\omega}_3 = (\gamma+\varepsilon)\nabla^2\omega_3 + 2\alpha e_{3jk}\partial_j u_k - 4\alpha\omega_3 + \rho m_3, \tag{10.b}$$

where $\vec{u}$ is the displacement vector, $\vec{\omega}$ the rotation vector, $\rho$ the density, $j$ the microinertia, $\lambda, \mu$ the Lamé constants, and $\alpha, \gamma, \varepsilon$ the new elastic parameters. The following connection between the elastic parameters is adopted according to [8,9]:

$$\beta = \gamma - \varepsilon, \qquad \mu_c = \mu - \alpha, \qquad \gamma + \varepsilon = 2\mu_c l^2, \tag{11}$$

where $l$ is the characteristic length of the microstructure.

**Results and discussion**

Dispersion relation for a half-space:The general solution of (10) is sought in the form

$$\vec{u}(x,z,t)=\hat{\vec{u}}(z)e^{i(kx-\omega t)}, \qquad \vec{\omega}(x,z,t)=\hat{\vec{\omega}}(z)e^{i(kx-\omega t)}, \tag{12}$$

where $\hat{\vec{u}}$ and $\hat{\vec{\omega}}$ are the displacement and moment amplitudes (vector quantities). Substitution of (12) into (10), neglecting body forces, gives a system of ordinary differential equations with complex coefficients:

$$\hat{\varphi}''-\nu_1^2\hat{\varphi}=0, \qquad \hat{\psi}''-\nu_2^2\hat{\psi}+\frac{2i\alpha k}{\mu+\alpha}\hat{\omega}_3=0, \qquad \hat{\omega}_{3''}-\nu_3^2\hat{\omega}_3+\frac{2\alpha}{\gamma+\varepsilon}\left(\hat{\psi}''+k^2\hat{\psi}\right)=0, \tag{13}$$

$$\nu_1^2=k^2-\frac{\rho\omega^2}{\lambda+2\mu}, \qquad \nu_2^2=k^2-\frac{\rho\omega^2}{\mu+\alpha}, \qquad \nu_3^2=k^2+\frac{4\alpha-I\omega^2}{\gamma+\varepsilon}. \tag{14}$$

Three types of solutions can be extracted from (3.47)–(3.49), depending on the boundary conditions: (i) Rayleigh waves in the half-space decaying with depth; (ii) Lamb waves — symmetric and antisymmetric modes of a plate, independent of the depth; (iii) Stoneley waves at the contact of two half-spaces under various (rigid or smooth) contact conditions. Thus the system is a universal framework for the wave dynamics of Cosserat media.

In a non-axisymmetric medium the surface wave (classical Rayleigh wave) is dispersive, in contrast to the classical medium. For such waves two dispersion branches exist on the "phase velocity–frequency" plane: a lower ("acoustic") and an upper ("optical") branch. As the frequency increases, the phase velocity of the lower branch decreases, while the phase velocity associated with the upper branch increases with frequency. Over the whole frequency range the phase velocity of the surface wave exceeds the phase velocity of the bulk displacement wave (Fig. 1). The wave propagates in the horizontal direction and decays with depth, which was confirmed by the computations. The boundary conditions of a stress-free surface (vanishing force and couple stresses) lead to a self-consistent complex dispersion equation (plane-strain state).

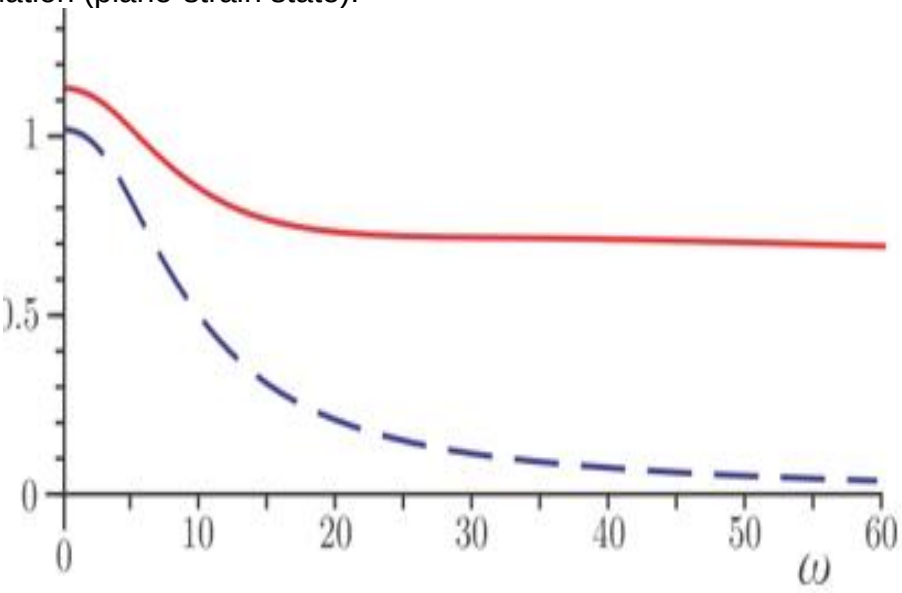



**Fig. 1. Dependence of the surface-wave velocity (red line) and of the bulk displacement-wave phase velocity (blue line) on frequency.**

Comparison with the classical medium: The numerical results show that the normal stress moment in the Cosserat medium is approximately 24 % larger than in the classical medium. The trajectory of the displacement amplitude is elliptical, as in the classical case; similar results were obtained for the Cosserat viscoelastic material.

## Conclusions

1. The equations of motion of a viscoelastic Cosserat medium are reduced by the Green–Lamb transformation to a second-order system of ordinary differential equations with complex coefficients; its general solution is constructed.

2. A dispersion equation for multilayered Cosserat structures is derived; the transcendental complex equation is solved numerically by the Müller–Laplace method.

3. The surface Rayleigh wave in a Cosserat half-space is dispersive and possesses two branches (acoustic and optical). Its phase velocity exceeds the bulk displacement-wave velocity in the whole frequency range, and the wave decays exponentially with depth.

4. From the same system, Lamb (symmetric/antisymmetric) and Stoneley wave solutions are obtained under various contact conditions.

5. The normal stress moment in the Cosserat medium exceeds the classical value by about 24 %; particle trajectories remain elliptical.

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